

# The structure of decomposable lattices determined by their prime ideals

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**Abstract:** A distributive lattice  $L$  with minimum element  $0$  is called decomposable if  $a$  and  $b$  are not comparable elements in  $L$  then there exist  $\bar{a}, \bar{b} \in L$  such that  $a = \bar{a} \vee (a \wedge b)$ ,  $b = \bar{b} \vee (a \wedge b)$  and  $\bar{a} \wedge \bar{b} = 0$ . The main purpose of this paper is to study the structure of decomposable lattices determined by their prime ideals. The properties for five special decomposable lattices are derived.

**Key Words:** decomposable lattice, prime ideal, minimal prime ideal, special ideal.

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## 1. Introduction and main results

Following [10], a decomposable lattice is a distributive lattice  $L$  with minimum element  $0$  such that for any  $a, b \in L$ , if  $a$  and  $b$  are not comparable elements in  $L$ , then there exist  $\bar{a}, \bar{b} \in L$  such that  $a = \bar{a} \vee (a \wedge b)$ ,  $b = \bar{b} \vee (a \wedge b)$  and  $\bar{a} \wedge \bar{b} = 0$ . The idea of decomposable lattice is originated from that of normal lattices and relatively

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normal lattices (see e.g. [5,6,9,11,12,13]). We have described prime ideals, minimal prime ideals and special ideals of a decomposable lattice in [10].

There are lots of decomposable lattices. In fact, it includes all distributive and strongly projectable lattices, all the positive cones of complete and compactly generated lattice-ordered groups, and the lattices of ideals of some arithmetical rings. In [10] the authors first established respectively a series of characterizations of prime ideals, minimal prime ideals and special ideals of a decomposable lattice and then investigated the relationship among them. All these characterizations will be our main technical tool for the further study of the structure of such lattices. In the present paper, we shall apply the results in [10] to study the structure of decomposable lattices determined by their prime ideals. All results in this paper are purely lattice-theoretic extension of some results of lattice-ordered groups (see e.g. [2,3,4,7]).

Here is a brief outline of the article. We simultaneously state the main results.

In Section 2, we simply review some basic definitions and introduce some notations for the classes of decomposable lattices satisfy some special conditions.

In Section 3, we investigate decomposable lattices in which every prime ideal contains at most  $n$  minimal prime ideals. By using the results in [10] and the pigeonhole principle, we shall prove that the every prime ideal of a decomposable lattice  $L$  contains at most  $n$  minimal prime ideals if and only if for any  $n+1$  mutually disjoint elements  $a_1, a_2, \dots, a_n, a_{n+1}$ ,  $L = a_1^\perp \vee a_2^\perp \vee \dots \vee a_n^\perp \vee a_{n+1}^\perp$ .

In Section 4, we investigate decomposable lattices with basis and prove that the following conditions are equivalent for a decomposable lattice  $L$ : (1)  $L$  has a basis; (2) for any  $0 < x \in L$ , there exists a basic element  $a$  such that  $x \geq a$ ; (3)  $P(L)$  is atomic; (4) for any  $A \in P(L) \setminus \{L\}$ ,  $A = \bigcap \{P \in P(L) \mid P \supseteq A, \text{ and } P \text{ is a maximal polar ideal of } L\}$ ; (5)  $\bigcap \{P \in P(L) \mid P \text{ is a maximal polar ideal of } L\} = 0$ . As an application of this result, we further investigate decomposable lattices with finite basis and prove that the following conditions are equivalent for a decomposable lattice  $L$ : (1)  $L$  has a finite basis; (2)  $P(L)$  is finite; (3)  $P(L)$  satisfies *DCC*.

In Section 5, we investigate decomposable lattices with compact property in the sense of Bigard-Conrad-Wolfenstein [1] and prove that the following conditions are equivalent for a decomposable lattice  $L$ : (1)  $L$  is compact; (2)  $L$  is discrete and each minimal prime ideal of  $L$  is a polar; (3) for any  $M \in \text{MinSpe}(L)$ , there exists

an atom  $a$  of  $L$  such that  $a \notin M$ ; (4) each ultrafilter of  $L$  is principal. This result is purely lattice-theoretic extension of the corresponding result of lattice-ordered groups. We apply the result to further investigate the relationship between compact property and countably compact property.

In Section 6, we investigate decomposable lattices  $L$  in which  $V(L)$ ,  $P(L)$  and  $Ide(L)$  satisfy  $DCC$ , respectively, and prove that  $Ide(L)$  satisfies  $DCC$  if and only if  $V(L)$  and  $P(L)$  satisfy  $DCC$ , respectively.

In last section, we investigate decomposable lattices in which each nonzero element has only finitely many values and decomposable lattices in which each disjoint subset with upper bound is finite. Moreover, we also investigate consistency of decomposable lattices and establish a simply connection between consistency and projectivity in the category of decomposable lattices in which each nonzero element has only finitely many values.

## 2. Preliminaries and notations

In this section, we simply review some basic definitions and some well-known results. The readers are referred to [8] for the general theory of lattices.

Throughout this paper, we consider lattices  $L$  with minimum element 0, denote by  $\mathbb{DL}$  the class of decomposable lattices and use " $\subset$ " and " $\supset$ " to denote proper set-inclusion.

A lattice  $L$  is called distributive if  $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$  for any  $a, b, c \in L$ . A nonempty subset  $I$  in a lattice  $L$  is called an ideal of  $L$  if  $a \vee b \in I$  for any  $a, b \in I$  and  $a \geq x \in L$  implies that  $x \in I$ . We denote by  $Ide(L)$  the set of all ideals of  $L$ . In particular, if  $a \in L$  then  $(a) = \{x \in L \mid x \leq a\}$  is called the principal ideal of  $L$  generated by  $a$ . A direct computation shows that if  $L \in \mathbb{DL}$  then  $Ide(L)$  is a distributive lattice by the rule:  $I \wedge J = I \cap J$  and  $I \vee J = \{a \vee b \mid a \in I, b \in J\}$  for any  $I, J \in Ide(L)$ .

An ideal  $P$  in a lattice  $L$  is called prime if  $P \neq L$  and  $a \wedge b \in P$  implies that either  $a \in P$  or  $b \in P$ , where  $a, b \in L$ . By Zorn's Lemma, each prime ideal contains a minimal prime ideal. We denote by  $Spe(L)$  and  $MinSpe(L)$  respectively the set of all prime ideals of  $L$  and the set of all minimal prime ideals of  $L$ .

Let  $L$  be a lattice. For any  $0 < x \in L$ , by Zorn's Lemma, there exists a maximal ideal of  $L$  with respect to not containing  $x$ , denoted  $M$ ,  $M$  is called a regular ideal

and is the value of  $x$ . In general,  $a$  need not have a unique value. We denote by  $Val(x)$  the set of all values of  $x$ . If  $M$  is the unique value of  $x$ ,  $M$  or  $x$  is called special. We denote by  $V(L)$  and  $S(L)$  respectively the set of all values of  $L$  and the set of all special values of  $L$ . Clearly,  $S(L) \subseteq V(L)$ . Observe that the following conditions are equivalent: (1)  $M \in V(L)$ ; (2)  $M$  is meet-irreducible, i.e., if  $\bigcap_{\lambda \in \Lambda} I_\lambda = M$ , where  $\{I_\lambda\}_{\lambda \in \Lambda} \subseteq Ide(L)$ , then  $I_\lambda = M$  for some  $\lambda$ ; (3)  $M \subset M^* = \bigcap \{I \in Ide(L) \mid I \supset M\}$ ; (4)  $M \in Val(x)$ , where  $x \in M^* \setminus M$ .

For a lattice  $L$  and  $\emptyset \neq A \subseteq L$ , we write  $A^\perp = \{x \in L \mid x \wedge a = 0 \text{ for any } a \in A\}$ .  $A^\perp$  is called the polar of  $A$ , and define  $(A^\perp)^\perp = A^{\perp\perp}$ .  $P \in Ide(L)$  is called polar if  $P = A^\perp$  for some  $\emptyset \neq A \subseteq L$ . Clearly,  $P \in Ide(L)$  is polar if and only if  $P = P^{\perp\perp}$ . We denote by  $P(L)$  the set of all polar ideals of  $L$ .

An element  $a$  in a lattice  $L$  is called a basic element if  $a > 0$  and  $[a]$  is totally ordered. A nonempty subset  $\{a_\lambda\}_{\lambda \in \Lambda}$  of  $L$  is called a basis if this set is a maximal disjoint subset in  $L$  and each element is a basic element.

Let  $L$  be a lattice and  $\emptyset \neq F \subseteq L$ . A nonempty subset  $F$  of  $L$  is called a filter of  $L$  if the following conditions are satisfied: (1)  $0 \notin F$ ; (2) for any  $a, b \in F$ ,  $a \wedge b \in F$ ; (3) if  $x \in L$  and  $x \geq a \in F$  implies  $x \in F$ . By Zorn's Lemma, each filter  $F$  of  $L$  must be contained in a maximal filter  $U$  of  $L$ , and  $U$  is called an ultrafilter of  $L$ . A filter  $F$  of  $L$  is called principal if  $F = \{x \in L \mid x \geq a\}$  for some  $a \in L$ .

In this article,  $L$  will be always a lattice unless otherwise stated. For convenience, we use the following notations to denote classes of special lattices.

$$\mathbb{A} = \{L \mid \text{every prime ideal of } L \text{ is minimal}\}.$$

$$\mathbb{B} = \{L \mid \text{every prime ideal of } L \text{ contains a unique minimal prime ideals}\}.$$

$$\mathbb{B}_n = \{L \mid \text{every prime ideal of } L \text{ contains at most } n \text{ minimal prime ideals}\}.$$

$$\mathbb{B}_\omega = \{L \mid \text{every prime ideal of } L \text{ contains at most finitely many minimal prime ideals}\}.$$

$$\mathbb{C} = \{L \mid L \text{ is compact}\}.$$

$$\mathbb{C}_\omega = \{L \mid L \text{ is countably compact}\}.$$

$$\mathbb{D} = \{L \mid Ide(L) \text{ satisfies } DCC\}.$$

$$\mathbb{E} = \{L \mid V(L) \text{ satisfies } DCC\}.$$

$$\mathbb{F} = \{L \mid \text{every disjoint subset of } L \text{ with upper bound is finite}\}.$$

$\mathbb{F}_v = \{L \mid \text{every nonzero element of } L \text{ has only finitely many values} \}$ .

$\mathbb{S} = \{L \mid L \text{ has a basis} \}$ .

$\mathbb{S}_\omega = \{L \mid L \text{ has a finite basis} \}$ .

$\mathbb{T} = \{L \mid L \text{ is projectable, i.e., for any } a \in L, L = a^{\perp\perp} \vee a^\perp \}$ .

### 3. $\mathbb{B}$ and $\mathbb{B}_n$

In this section, we shall investigate decomposable lattices in which every prime ideal contains at most  $n$  minimal prime ideals. By using the results in [10] and the pigeonhole principle, we shall establish explicit characterizations for the class of such lattices.

First, we need the following two lemmas ([10], Lemma 4.2 and Lemma 5.7).

**Lemma 3.1.** Let  $L \in \mathbb{DL}$ . If  $P \in \text{Spe}(L)$  then

$$\bigcup \{a^\perp \mid a \in L \setminus P\} = \bigcap \{M \in \text{MinSpe}(L) \mid M \subseteq P\}.$$

**Lemma 3.2.** Let  $L \in \mathbb{DL}$ . If  $Q_1, Q_2, \dots, Q_n$  are mutually incomparable prime ideals of  $L$  and  $a \notin Q_i$  for  $i = 1, 2, \dots, n$ , then there exist  $a_i \in (\bigcap_{j \neq i} Q_j) \setminus Q_i$  such that  $0 < a_i < a$  for  $i = 1, 2, \dots, n$  and  $a_i \wedge a_j = 0$  for  $i \neq j$ .

We now state and prove the main result of this section.

**Theorem 3.3.** Let  $L \in \mathbb{DL}$ . The following conditions are equivalent:

- (1)  $L \in \mathbb{B}_n$ .
- (2) For any  $n + 1$  distinct minimal prime ideals  $M_1, M_2, \dots, M_n, M_{n+1}$  of  $L$ ,

$$L = M_1 \vee M_2 \vee \dots \vee M_n \vee M_{n+1}.$$

- (3) For any  $n + 1$  mutually incomparable values  $Q_1, Q_2, \dots, Q_n, Q_{n+1}$  of  $L$ ,

$$L = Q_1 \vee Q_2 \vee \dots \vee Q_n \vee Q_{n+1}.$$

- (4) For any  $n + 1$  mutually incomparable prime ideals  $P_1, P_2, \dots, P_n, P_{n+1}$  of  $L$ ,

$$L = P_1 \vee P_2 \vee \dots \vee P_n \vee P_{n+1}.$$

- (5) For any  $n + 1$  mutually disjoint elements  $a_1, a_2, \dots, a_n, a_{n+1}$ ,

$$L = a_1^\perp \vee a_2^\perp \vee \cdots \vee a_n^\perp \vee a_{n+1}^\perp.$$

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) is clear. It suffices to show (1) $\Leftrightarrow$ (5).

(1) $\Rightarrow$ (5) Assume that there exist  $n+1$  mutually disjoint elements  $a_1, a_2, \dots, a_n, a_{n+1}$  in  $L$  such that  $\bigvee_{i=1}^{n+1} a_i^\perp \subset L$ . Pick  $x \in L \setminus (\bigvee_{i=1}^{n+1} a_i^\perp)$ . Then there exists some  $M \in Val(x)$  such that  $M \supseteq \bigvee_{i=1}^{n+1} a_i^\perp$ . By Lemma 3.1, we have

$$\bigcap \{P \in MinSpe(L) \mid P \subseteq M\} = \bigcup \{h^\perp \mid h \notin M\}.$$

Now, write  $V_0 = \bigcup \{h^\perp \mid h \notin M\}$ . We claim that  $a_i \notin V_0$  for any  $i$  ( $i = 1, 2, \dots, n, n+1$ ). Assume that  $a_i \in V_0$  for some  $i$ . Then  $a_i \in h^\perp$  for some  $h \notin M$ . Thus  $h \in a_i^\perp \subseteq M$ , a contradiction. Since  $L \in \mathbb{B}_n$ ,  $M$  contains at most  $n$  minimal prime ideals, write  $Q_1, Q_2, \dots, Q_n$ . Then

$$V_0 = \bigcap_{j=1}^n Q_j = \bigcup \{h^\perp \mid h \notin M\}.$$

By the pigeonhole principle, there exists a minimal prime ideal, denoted  $Q_i$ , contained in  $M$ , which does not contain two of the elements of the set  $\{a_1, a_2, \dots, a_n, a_{n+1}\}$ . Since  $Q_i$  is prime, this is not impossible. So  $L = a_1^\perp \vee a_2^\perp \vee \cdots \vee a_n^\perp \vee a_{n+1}^\perp$ .

(5) $\Rightarrow$ (1) Assume that there exists some  $Q \in Spe(L)$  such that  $Q$  contains  $n+1$  distinct minimal prime ideals of  $L$ , write  $Q_1, Q_2, \dots, Q_n, Q_{n+1}$ . Clearly, they are mutually incomparable. So, by Lemma 3.2, there exist

$$a_i \in \left( \bigcap_{1 \leq j \neq i \leq n+1} Q_j \right) \setminus Q_i, \text{ where } i, j = 1, 2, \dots, n, n+1,$$

such that  $a_i \wedge a_j = 0$  for  $i \neq j$ . Now, for any  $i$  ( $i = 1, 2, \dots, n, n+1$ ), since  $a_i \notin Q_i$ ,  $a_i^\perp \subseteq Q_i$ . So

$$Q \supseteq Q_1 \vee Q_2 \vee \cdots \vee Q_n \vee Q_{n+1} \supseteq a_1^\perp \vee a_2^\perp \vee \cdots \vee a_n^\perp \vee a_{n+1}^\perp = L,$$

which is a contradiction. Therefore  $L \in \mathbb{B}_n$ .  $\square$

As a direct result of Theorem 3.3, we have

**Corollary 3.4.** Let  $L \in \mathbb{DL}$ . If for any  $M \in MinSpe(L)$ ,  $L = M \vee M^\perp$ , then  $L \in \mathbb{B}$ .

#### 4. $\mathbb{S}$ and $\mathbb{S}_\omega$

In this section, we investigate decomposable lattices with basis or finite basis. We shall establish a series of characterizations for them.

Recall that an element  $a$  in a lattice  $L$  is called a basic element if  $a > 0$  and  $(a]$  is totally ordered. A nonempty subset  $\{a_\lambda\}_{\lambda \in \Lambda}$  of  $L$  is called a basis if this set is a maximal disjoint subset in  $L$  and each element is a basic element.

The following two lemmas are well known ([10], Theorem 4.6 and Theorem 4.3).

**Lemma 4.1.** Let  $L \in \mathbb{DL}$  and  $0 \neq I \in Ide(L)$ . The following conditions are equivalent:

- (1)  $I$  is totally ordered.
- (2) For any  $0 < a \in I$ ,  $a^\perp = I^\perp$ .
- (3)  $I^\perp \in Spe(L)$ .
- (4)  $I^\perp \in MinSpe(L)$ .
- (5)  $I^{\perp\perp}$  is a maximal totally ordered ideal of  $L$ .
- (6)  $I^{\perp\perp}$  is a minimal polar ideal of  $L$ .
- (7)  $I^\perp$  is a maximal polar ideal of  $L$ .
- (8) For any  $0 < a \in I$ ,  $a$  is special.

**Lemma 4.2.** Let  $L \in \mathbb{DL}$  and  $P \in Spe(L)$ . The following conditions are equivalent:

- (1)  $P \in MinSpe(L)$ .
- (2)  $P = \bigcup \{a^\perp \mid a \notin P\}$ .
- (3) For any  $x \in P$ ,  $x^\perp \not\subseteq P$ .

We shall first apply Lemma 4.1 and Lemma 4.2 to establish characterizations of decomposable lattices with basis.

**Theorem 4.3.** Let  $L \in \mathbb{DL}$ . The following conditions are equivalent:

- (1)  $L \in \mathbb{S}$ .
- (2) For any  $0 < x \in L$ , there exists a basic element  $a$  such that  $x \geq a$ .
- (3)  $P(L)$  is atomic, i.e., for any  $A \in P(L) \setminus \{0, L\}$ , there exists a minimal polar ideal  $B$  of  $L$  such that  $A \supseteq B$ .
- (4) For any  $A \in P(L) \setminus \{L\}$ ,  $A = \bigcap \{P \in P(L) \mid P \supseteq A, \text{ and } P \text{ is a maximal polar ideal of } L\}$ .
- (5)  $\bigcap \{P \in P(L) \mid P \text{ is a maximal polar ideal of } L\} = 0$ .

**Proof.** (1) $\Rightarrow$ (2) Let  $\{a_\lambda\}_{\lambda \in \Lambda}$  be a basis of  $L$ . Now, for any  $0 < x \in L$ , since  $\{a_\lambda\}_{\lambda \in \Lambda}$  is a maximal disjoint subset in  $L$ , there exists some  $\lambda \in \Lambda$  such that  $x \wedge a_\lambda > 0$ . Since  $a_\lambda$  is a basic element,  $x \wedge a_\lambda$  is clearly a basic element and  $x \geq x \wedge a_\lambda$ .

(2) $\Rightarrow$ (3) Given any  $A \in P(L) \setminus \{0, L\}$ , pick  $0 < x \in A$ . By (2), there exists a basic element  $a$  such that  $x \geq a$ . Then  $a^{\perp\perp} \subseteq x^{\perp\perp} \subseteq A$ . By Lemma 4.1,  $a^{\perp\perp}$  is a minimal polar ideal of  $L$ . So  $P(L)$  is atomic.

(3) $\Rightarrow$ (4) Since the map  $P \rightarrow P^\perp$  for any  $P \in P(L)$  is a dual isomorphism of lattices, by (3), for any  $A \in P(L) \setminus \{L\}$ , there exists a maximal polar ideal  $P$  of  $L$  such that  $P \supseteq A$ . Consider the set

$$\Omega = \{P \in P(L) \mid P \supseteq A, \text{ and } P \text{ is a maximal polar ideal of } L\}.$$

Clearly,  $A \subseteq \bigcap_{P \in \Omega} P$ . If  $0 < x \notin A = A^{\perp\perp}$ , then there exists some  $0 < a \in A^\perp$  such that  $x \wedge a > 0$ . By (3), there exists a maximal polar ideal  $P$  such that  $(x \wedge a)^\perp \subseteq P$ . So  $A = A^{\perp\perp} \subseteq a^\perp \subseteq (x \wedge a)^\perp \subseteq P$ . By Lemma 4.1,  $P$  is a maximal polar ideal implies that  $P \in \text{MinSpe}(L)$ . So, by Lemma 4.2,  $x \wedge a \notin P$ , and hence  $x \notin P$ . Therefore  $A = \bigcap_{P \in \Omega} P$ .

(4) $\Rightarrow$ (5) Suppose that  $\bigcap\{P \in P(L) \mid P \text{ is a maximal polar ideal of } L\} \neq 0$ . Pick  $0 < a \in \bigcap\{P \in P(L) \mid P \text{ is a maximal polar ideal of } L\}$ . By (4),  $a^\perp \subseteq P$  for some maximal polar ideal  $P$  of  $L$ . Again, by Lemma 4.1,  $P \in \text{MinSpe}(L)$ . But  $a \in P$  and  $a^\perp \subseteq P$ , which contradicts Lemma 4.2. So  $\bigcap\{P \in P(L) \mid P \text{ is a maximal polar ideal of } L\} = 0$ .

(5) $\Rightarrow$ (1) Let  $\bigcap_{\lambda \in \Lambda} P_\lambda = 0$ , where each  $P_\lambda$  is a maximal polar ideal of  $L$ . Then each  $P_\lambda^\perp$  is a minimal polar ideal of  $L$ . By Lemma 4.1,  $P_\lambda^\perp$  is totally ordered. Now, pick  $0 < a_\lambda \in P_\lambda^\perp$  for any  $\lambda \in \Lambda$ . Clearly, each  $a_\lambda$  is a basic element of  $L$ . Set  $A = \{a_\lambda \mid \lambda \in \Lambda\}$ . We shall show that  $A$  is a basis of  $L$ . For any  $\alpha, \beta \in \Lambda$  with  $\alpha \neq \beta$ ,  $P_\alpha^\perp$  and  $P_\beta^\perp$  are both minimal polar ideals of  $L$ ,  $a_\alpha \wedge a_\beta \in P_\alpha^\perp \cap P_\beta^\perp = 0$ . In addition, if  $x \wedge a_\lambda = 0$  for any  $\lambda \in \Lambda$ , where  $x \in L$ , then  $x \in a_\lambda^\perp = P_\lambda^{\perp\perp} = P_\lambda$  for any  $\lambda \in \Lambda$ . Thus  $x \in \bigcap_{\lambda \in \Lambda} P_\lambda = 0$ . So  $A = \{a_\lambda \mid \lambda \in \Lambda\}$  is a basis of  $L$ . Therefore  $L \in \mathbb{S}$ .  $\square$

As an application of Theorem 4.3, we have

**Corollary 4.4.** Let  $L \in \mathbb{DL}$ . If every minimal prime ideal of  $L$  is a polar ideal, i.e.,  $\text{MinSpe}(L) \subseteq P(L)$ , then  $L \in \mathbb{S}$ .



**Proof.** Given any  $M \in \text{MinSpe}(L)$ , there exists  $\emptyset \neq A \subseteq L$  such that  $M = A^\perp$ . Clearly,  $A \neq \{0\}$ . Pick  $0 < a \in A$ . Then  $M = A^\perp \subseteq a^\perp$ .

First, we claim that  $M = a^\perp$ . Assume that  $a^\perp \supset M$ . Then  $a \notin M$ . Pick  $0 < b \in a^\perp \setminus M$ . Since  $a \wedge b = 0 \in M$  and  $M$  is prime, this means that either  $a \in M$  or  $b \in M$ , a contradiction. So  $M = a^\perp$ .

Second, we show that  $a$  is a basic element. Otherwise, there exist  $0 < a_1, a_2 < a$  such that  $a_1 \wedge a_2 = 0 \in M = a^\perp$ . So either  $a_1 \in a^\perp$  or  $a_2 \in a^\perp$ , which implies that either  $a_1 = 0$  or  $a_2 = 0$ , a contradiction.

Finally, we show that  $L$  has a basis. Assume that  $L$  has no basis. By Theorem 4.3, there exists  $0 < x \in L$  such that  $x$  does not exceed any basic elements. Let  $Q_x$  be a value of  $x$ . Since every prime ideal of  $L$  contains at least a minimal prime ideal. Without loss of generality, suppose that  $M \subseteq Q_x$ . Clearly,  $x \notin M = a^\perp$ , so  $x \wedge a > 0$ . Notice that  $x \wedge a$  is a basic element and  $x \geq x \wedge a$ , a contradiction. Therefore  $L \in \mathbb{S}$ .  $\square$

For a decomposable lattice  $L$  and  $M \in V(L)$ ,  $M$  is called essential if there exists  $0 < x \in L$  such that for any  $G_\lambda \in \text{Val}(x)$ ,  $G_\lambda \subseteq M$ . We denote by  $E(L)$  the set of all essential values of  $L$ . Clearly,  $S(L) \subseteq E(L)$ . Write

$$\text{Rad}(L) = \bigcap E(L),$$

and  $\text{Rad}(L)$  is called the radical of  $L$ .

In the following, we shall use Theorem 4.3 to establish a connection between decomposable lattices with basis and  $\text{Rad}(L) = 0$ .

**Corollary 4.5.** Let  $L \in \mathbb{DL}$  and  $L \in \mathbb{B}_\omega$ . The following conditions are equivalent:

- (1)  $L \in \mathbb{S}$ .
- (2)  $\bigcap S(L) = 0$ .
- (3)  $\text{Rad}(L) = 0$ .

**Proof.** (1) $\Rightarrow$ (2) Let  $\{s_\lambda \mid \lambda \in \Lambda\}$  be a basis of  $L$ . Assume that  $\bigcap S(L) \neq 0$ . Pick  $0 < x \in \bigcap S(L)$ . By Theorem 4.3, there exists some  $\lambda \in \Lambda$  such that  $x \geq s_\lambda$ . Since  $s_\lambda$  is special, let  $Q_\lambda$  be the unique value of  $s_\lambda$ , then  $s_\lambda \notin Q_\lambda$ , so that  $s_\lambda \notin \bigcap S(L)$ , which contradicts the fact that  $x \in \bigcap S(L)$  implies  $s_\lambda \in \bigcap S(L)$ .

(2) $\Rightarrow$ (3) Since  $S(L) \subseteq E(L)$ ,  $\text{Rad}(L) \subseteq \bigcap S(L) = 0$ . So  $\text{Rad}(L) = 0$ .

(3) $\Rightarrow$ (1) Given any  $0 < a \in L$ , since  $\text{Rad}(L) = 0$ , there exists some  $M \in E(L)$  such that  $a \notin M$ . So there exists  $Q \in \text{Val}(a)$  such that  $Q \supseteq M$ . Since  $M \in E(L)$ ,  $Q \in E(L)$ .

Now, we claim that there exists a basic element  $s$  of  $L$  such that  $a \geq s$ . Since  $Q$  is essential, there exists  $0 < f \in L$  such that all the values of  $f$  are contained in  $Q$ . Set  $g = a \wedge f$ . Clearly, all the values of  $g$  are also contained in  $Q$ . For convenience, we may suppose that  $f < a$ . Since  $L \in \mathbb{B}_\omega$ , there exists a positive integer  $k$  such that  $Q$  contains at most  $k$  minimal prime ideals. Now, if  $f$  does not exceed a basic element, then there exists a disjoint subset  $\{f_1, f_2, \dots, f_m\}$  of  $L$  with upper bound  $f$  and satisfies  $m > k$ . Let  $Q_i$  be a value of  $f_i$  for  $i = 1, 2, \dots, m$ . Clearly,  $Q_i \parallel Q_j$  for  $i \neq j$  and each  $Q_i \subseteq Q$ , a contradiction. So  $a$  must exceed a basic element of  $L$ . Therefore  $L \in \mathbb{S}$ .  $\square$

In order to establish characterizations of decomposable lattices with finite basis, we need the following lemma.

**Lemma 4.6.** Let  $L \in \mathbb{DL}$  and let  $\{a_1, a_2, \dots, a_n\}$  be a basis of  $L$  and  $a_i^{\perp\perp} = A_i$  for  $1 \leq i \leq n$ . Then  $(\bigvee_{i \in \Delta} A_i)^\perp = (\bigvee_{i \in N \setminus \Delta} A_i)^{\perp\perp}$ , where  $N = \{1, 2, \dots, n\}$  and  $\Delta \subseteq N$ .

**Proof.** We divide the proof into two steps.

**Step 1.** If  $\Delta = \emptyset$  then  $\bigvee_{i \in \Delta} A_i = 0$ , we are done.

**Step 2.** If  $\Delta \neq \emptyset$  then  $a_i \wedge a_j = 0$  for any  $i \neq j$ . Thus  $a_i \in a_j^\perp$ , and hence  $a_j^{\perp\perp} \subseteq a_i^\perp$ , so that  $a_i^{\perp\perp} \cap a_j^{\perp\perp} = 0$ , i.e.,  $A_i \cap A_j = 0$ . So

$$\bigvee_{i \in \Delta} A_i \subseteq (\bigvee_{i \in N \setminus \Delta} A_i)^\perp \Rightarrow (\bigvee_{i \in N \setminus \Delta} A_i)^{\perp\perp} \subseteq (\bigvee_{i \in \Delta} A_i)^\perp.$$

On the other hand,  $\{a_1, a_2, \dots, a_n\}$  is a basis of  $L$ , so  $(\bigvee_{i \in N} A_i)^\perp = 0$ . Thus

$$(\bigvee_{i \in \Delta} A_i)^\perp \cap (\bigvee_{i \in N \setminus \Delta} A_i)^\perp = 0,$$

so that  $(\bigvee_{i \in \Delta} A_i)^\perp \subseteq (\bigvee_{i \in N \setminus \Delta} A_i)^{\perp\perp}$ . Therefore  $(\bigvee_{i \in \Delta} A_i)^\perp = (\bigvee_{i \in N \setminus \Delta} A_i)^{\perp\perp}$ .  $\square$

**Theorem 4.7.** Let  $L \in \mathbb{DL}$ . The following conditions are equivalent:

(1)  $L \in \mathbb{S}_\omega$ .

(2)  $P(L)$  is finite.

(3)  $P(L)$  satisfies *DCC*.

**Proof.** (1) $\Rightarrow$ (2) Let  $\{a_1, a_2, \dots, a_n\}$  be a finite basis of  $L$ . Set  $A_i = a_i^{\perp\perp}$  for  $1 \leq i \leq n$ . Then each  $A_i$  is a minimal polar ideal of  $L$  by Lemma 4.1. So, for any  $P \in P(L)$ , either  $P \cap A_i = 0$  or  $P \cap A_i = A_i$ . Set  $\Delta = \{1, 2, \dots, n\}$ , and

$$\Delta_1 = \{i \in \Delta \mid P \cap A_i = 0\} \quad \text{and} \quad \Delta_2 = \{i \in \Delta \mid P \cap A_i = A_i\}.$$

So

$$\bigvee_{i \in \Delta_1} A_i \subseteq P^\perp \quad \text{and} \quad \bigvee_{i \in \Delta_2} A_i \subseteq P.$$

Then

$$\bigvee_{i \in \Delta_1} A_i \subseteq P^\perp \subseteq \left( \bigvee_{i \in \Delta_2} A_i \right)^\perp \Rightarrow \left( \bigvee_{i \in \Delta_2} A_i \right)^{\perp\perp} \subseteq P^{\perp\perp} \subseteq \left( \bigvee_{i \in \Delta_2} A_i \right)^\perp.$$

By Lemma 4.6,  $P = P^{\perp\perp} = \left( \bigvee_{i \in \Delta_2} A_i \right)^{\perp\perp}$ . Therefore  $P(L)$  is finite.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1) We first show that  $L \in \mathbb{S}$ . Otherwise, there exists  $0 < a \in L$  such that  $a$  does not exceed any basic element. Hence there exist  $0 < a_0, b_0 < a$  such that  $a_0 \wedge b_0 = 0$ . For  $a_0$ ,  $a_0$  does not exceed any basic element. Hence there exist  $0 < a_1, b_1 < a_0$  such that  $a_1 \wedge b_1 = 0$ . Continuing this process, we can obtain an infinite descending chain of  $P(L)$  as follows:

$$a_0^\perp \supset a_1^\perp \supset \dots \supset a_n^\perp \supset \dots,$$

which contradicts the fact that  $P(L)$  satisfies *DCC*. So  $L$  must have a basis. Now, let  $\{s_\lambda \mid \lambda \in \Lambda\}$  be a basis of  $L$ . Assume that  $|\Lambda| = \infty$ . Then we can similarly obtain an infinite descending chain of  $P(L)$  as follows:

$$s_1^\perp \supset (s_1 \vee s_2)^\perp \supset \dots \supset (s_1 \vee s_2 \vee \dots \vee s_n)^\perp \supset \dots,$$

a contradiction. Therefore  $L \in \mathbb{S}_\omega$ .  $\square$

Recall that an element  $u$  in a lattice  $L$  is called a unit if  $u > 0$  and  $u \wedge x > 0$  for any  $0 < x \in L$ . Recall also that  $I \in Ide(L)$  is called large if  $I \cap J \neq 0$  for any  $0 \neq J \in Ide(L)$ . As an application of Theorem 4.7, we have

**Corollary 4.8.** Let  $L \in \mathbb{DL}$  and  $L \in \mathbb{B}_\omega$ . The following conditions are equivalent:

- (1)  $L \in \mathbb{S}_\omega$ .
- (2) There exists a large ideal  $I$  of  $L$  with the form  $I = \bigvee_{i=1}^n (a_i]$ , where each  $(a_i]$  is totally ordered.
- (3)  $L$  has a unit  $u$  and  $u$  has only finitely many values.

**Proof.** (1) $\Rightarrow$ (2) Let  $\{a_1, a_2, \dots, a_n\}$  be a finite basis of  $L$ . A direct computation shows that  $I = \bigvee_{i=1}^n (a_i]$  is a large ideal of  $L$ , and each  $(a_i]$  is clearly totally ordered.

(2) $\Rightarrow$ (3) For any  $0 < x \in L$ , since  $I$  is large and  $Id(L)$  is a distributive lattice, we then have

$$0 \neq (x] \cap I = (x] \cap \left( \bigvee_{i=1}^n (a_i] \right) = \bigvee_{i=1}^n ((x] \cap (a_i]).$$

So there exists some  $i$  ( $1 \leq i \leq n$ ) such that  $(x] \cap (a_i] \neq 0$ , i.e.,  $0 \neq x \wedge a_i \leq x$ . By Theorem 4.3,  $\{a_1, a_2, \dots, a_n\}$  is a finite basis of  $L$ . Now, let  $u = a_1 \vee a_2 \vee \dots \vee a_n$ . Clearly,  $u$  is a unit of  $L$ . Since each  $a_i$  is a basic element, each  $a_i$  must be special. Let  $Q_i$  be the unique value of  $a_i$ . Since  $\{a_1, a_2, \dots, a_n\}$  are mutually disjoint, we have

$$Val(u) = \bigcup_{i=1}^n Val(a_i) = \{Q_1, Q_2, \dots, Q_n\}.$$

So  $u$  has only finitely many values.

(3) $\Rightarrow$ (1) Let  $\{Q_1, Q_2, \dots, Q_k\}$  be the set of all values of  $u$ .

We first show that  $L \in \mathbb{F}$ . Otherwise, there exists  $0 < f \in L$  and an infinite disjoint subset of  $L$  with upper bound  $f$ , write  $\{a_i \in L \mid i \in I, |I| = \infty\}$ . Notice that  $u$  is a unit. Set  $b_i = u \wedge a_i$  for any  $i \in I$ . Then  $\{b_i \in L \mid i \in I, |I| = \infty\}$  is an infinite disjoint subset of  $L$  with upper bound  $u$ . Now, let  $M_i$  be a value of  $b_i$  for any  $i \in I$ . Since  $u \notin M_i$  for any  $i \in I$ ,  $M_i \subseteq Q_j$  for some  $j$  ( $j = 1, 2, \dots, k$ ). Notice that  $I$  is infinite, so that there exists some  $Q_j$  contains an infinite number of  $M_i$ , which contradicts  $L \in \mathbb{B}_\omega$ .

Since  $L \in \mathbb{B}_\omega$ , let  $Q_i$  contains  $n_i$  minimal prime ideals, and set  $m = \max\{n_i \mid 1 \leq i \leq k\}$ . We shall show that if the number of basic elements of  $L$  is  $n$ , then  $n \leq mk+1$ . Otherwise, there exists a disjoint subset  $\{x_1, x_2, \dots, x_t\}$  of  $L$  such that  $t > mk+1$ . Set  $y_i = x_i \wedge u$  for  $i = 1, 2, \dots, t$ . Then  $\{y_1, y_2, \dots, y_t\}$  is also a mutually disjoint

subset of  $L$  with upper bound  $u$ . Repeating the above process, we shall obtain that there exists some  $Q_i$  contains at least  $m + 1$  minimal prime ideals, a contradiction. Therefore  $L \in \mathbb{S}_\omega$ .  $\square$

## 5. $\mathbb{C}$ and $\mathbb{C}_\omega$

In this section, we shall first study the structure of decomposable lattices with compact property and then investigate the relationship between compact property and countably compact property.

Recall that a lattice  $L$  is called compact if  $\{a_\lambda\}_{\lambda \in \Lambda}$  is a nonempty subset of  $L$  and  $\bigwedge_{\lambda \in \Lambda} a_\lambda = 0$  then there exists a finite subset  $\{a_i\}_{i=1}^n$  of  $\{a_\lambda\}_{\lambda \in \Lambda}$  such that  $\bigwedge_{i=1}^n a_i = 0$ . Recall also that a lattice  $L$  is called discrete if every nonzero element of  $L$  exceeds an atom.

**Theorem 5.1.** Let  $L \in \mathbb{DL}$ . The following conditions are equivalent:

- (1)  $L \in \mathbb{C}$ .
- (2)  $L$  is discrete and each minimal prime ideal of  $L$  is a polar.
- (3) For any  $M \in \text{MinSpe}(L)$ , there exists an atom  $a$  of  $L$  such that  $a \notin M$ .
- (4) Each ultrafilter of  $L$  is principal.

**Proof.** (1) $\Rightarrow$ (2) We first show that  $L$  is discrete. Given any  $0 < x \in L$ , let  $\{a_\lambda\}_{\lambda \in \Lambda}$  be a maximal chain of  $L$  containing  $x$ . If  $\bigwedge_{\lambda \in \Lambda} a_\lambda = 0$ , then since  $L$  is compact, there exist a finite subset  $\{a_i\}_{i=1}^n$  of  $\{a_\lambda\}_{\lambda \in \Lambda}$  such that  $\bigwedge_{i=1}^n a_i = 0$ . Notice that  $a_1, a_2, \dots, a_n$  are mutually comparable and each  $a_\lambda > 0$ , this is clearly impossible. So, if we set  $a = \bigwedge_{\lambda \in \Lambda} a_\lambda$ , then  $a$  is an atom in  $L$  and  $x \geq a$ . Thus  $L$  is discrete.

We next show that every minimal prime ideal of  $L$  is a polar. Given any  $M \in \text{MinSpe}(L)$ ,  $M = \bigcup \{a^\perp \mid a \in L \setminus M\}$  by Lemma 4.2. Set  $K = L \setminus M$ . Notice that  $L \in \mathbb{DL}$  and  $M$  is prime, which implies that  $K$  is a chain of  $L$ . So  $K$  has an atom  $a$  such that  $M = a^\perp$ .

(2) $\Rightarrow$ (3) For any  $M \in \text{MinSpe}(L)$ , by (2),  $M \in P(L)$ . So  $M = A^\perp$  for some  $\emptyset \neq A \subseteq L$ . Now pick  $0 < x \in A$ . Since  $L$  is discrete, there exists an atom  $a$  in  $L$  such that  $x \geq a$ . We claim that  $a \notin M$ . Otherwise,  $a \in M = A^\perp$  implies  $a \in A \cap A^\perp = 0$ , a contradiction.

(3) $\Rightarrow$ (4) Let  $K$  be an ultrafilter of  $L$ . Then  $M = \bigcup\{a^\perp \mid a \in K\}$  is a minimal prime ideal of  $L$ . By (3), there exists an atom  $a \notin M$  such that  $a^\perp \subseteq M$ . Notice that  $a$  is an atom implies that  $a^\perp \in \text{MinSpe}(L)$  by Lemma 4.1, and hence  $M = a^\perp$ .

Now, it suffices to show that  $K = \{x \in L \mid x \geq a\}$ . For any  $x \in K$ , then  $x^\perp \subseteq M = a^\perp$ . So  $x \wedge a > 0$  for any  $x \in K$ . Since  $a$  is an atom,  $x \wedge a = a$ . Thus  $x \geq a$ . Conversely, given any  $x \in L$ , if  $x \geq a$  then since  $a \notin M$ , this means  $x \notin M$ , so that  $x \in K$ . Thus  $K = \{x \in L \mid x \geq a\}$ . So  $K$  is principal.

(4) $\Rightarrow$ (1) Let  $\{a_\lambda\}_{\lambda \in \Lambda}$  be a nonempty subset of  $L$ . Suppose that for any finite subset  $\{a_i\}_{i=1}^n$  of  $\{a_\lambda\}_{\lambda \in \Lambda}$ ,  $\bigwedge_{i=1}^n a_i \neq 0$ . Set  $a = \bigwedge_{i=1}^n a_i$ . Let  $Q \in \text{Val}(a)$  and  $M \in \text{MinSpe}(L)$  be such that  $M \subseteq Q$ . Then  $a \notin M$ . Since the set  $\{a_\lambda\}_{\lambda \in \Lambda}$  must be contained in an ultrafilter of  $L$ , write  $K$ . By (4),  $K = \{x \in L \mid x \geq b\}$  for some  $0 < b \in L$ . So  $\bigwedge_{\lambda \in \Lambda} a_\lambda \geq b > 0$ . Therefore  $L \in \mathbb{C}$ .  $\square$

We now apply Theorem 5.1 to establish the relationship between compact property and countably compact property.

Let us recall that a lattice  $L$  is called countably compact if  $\{a_i\}_{i=1}^\infty$  is a subset of  $L$  and  $\bigwedge_{i=1}^\infty a_i = 0$  then there exists a positive integer  $n$  such that  $\bigwedge_{i=1}^n a_i = 0$ .

**Theorem 5.2.** Let  $L \in \mathbb{DL}$ . If  $\text{MinSpe}(L) \subseteq P(L)$  then following conditions are equivalent:

- (1)  $L \in \mathbb{C}_\omega$ .
- (2) Every totally ordered ideal of  $L$  is countably compact.

**Proof.** (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) By way of contradiction. Assume that there exists a countable subset  $\{a_i\}_{i=1}^\infty$  of  $L$  such that  $\bigwedge_{i=1}^\infty a_i = 0$ , but  $\bigwedge_{i=1}^n a_i \neq 0$  for any positive integers  $n$ .

First, by Corollary 4.4,  $L$  has a basis. Let  $\{a_\lambda \mid \lambda \in \Lambda\}$  be a basis of  $L$ . We claim that for any  $\lambda \in \Lambda$  there exists some positive integer  $i$  such that  $a_\lambda \wedge a_i = 0$ . Otherwise, there exists some  $\mu \in \Lambda$  such that  $a_\mu \wedge a_i > 0$  for any positive integers  $i$ . Since

$$\bigwedge_{i=1}^\infty (a_\mu \wedge a_i) = a_\mu \wedge \left( \bigwedge_{i=1}^\infty a_i \right) = 0,$$

and

$$\{a_\mu \wedge a_i \mid i = 1, 2, \dots\} \subseteq (a_\mu],$$

by (2), there exists a positive integer  $n$  such that  $\bigwedge_{i=1}^n (a_\mu \wedge a_i) = 0$ . Notice that each  $a_\mu \wedge a_i > 0$  and  $\{a_\mu \wedge a_i \mid i = 1, 2, \dots, n\}$  is a finite chain, this is clearly impossible.

Second, since  $\{a_i\}_{i=1}^\infty$  is a  $\wedge$ -semilattice of  $L$ , this means that there exists an ultrafilter  $U$  of  $L$  such that  $\{a_i\}_{i=1}^\infty \subseteq U$ . So  $Q = \bigcup \{a^\perp \mid a \in U\}$  is a minimal prime ideal of  $L$ .

Finally, for any  $\lambda \in \Lambda$ , using the above result, there exists some positive integer  $i$  such that  $a_\lambda \wedge a_i = 0 \in Q$  and  $a_i \notin Q$ , then  $a_\lambda \in Q$  for any  $\lambda \in \Lambda$ . So  $\{a_\lambda \mid \lambda \in \Lambda\} \subseteq Q$ . But, by hypothesis,  $Q = A^\perp$  for some  $\emptyset \neq A \subseteq L$ , which implies that  $Q = a_\lambda^\perp$  for some basic element  $a_\lambda$  in  $L$ . This is clearly impossible. Therefore  $L \in \mathbb{C}_\omega$ .  $\square$

## 6. $\mathbb{D}$ and $\mathbb{E}$

In this section, we shall investigate decomposable lattices  $L$  in which  $Ide(L)$  satisfies  $DCC$  and prove that  $\mathbb{D} = \mathbb{E} \cap \mathbb{S}_\omega$ .

By a direct computation, we have

**Lemma 6.1.** Let  $L \in \mathbb{DL}$ . Then  $L \in \mathbb{E}$  if and only if  $Spe(L) = V(L)$ .

**Theorem 6.2.** Let  $L \in \mathbb{DL}$ . The following conditions are equivalent:

- (1)  $L \in \mathbb{D}$ .
- (2)  $V(L)$  and  $P(L)$  satisfy  $DCC$  respectively.
- (3)  $L \in \mathbb{E} \cap \mathbb{S}_\omega$ .

**Proof.** (1) $\Rightarrow$ (2) is automatic and (2) $\Leftrightarrow$ (3) is clear by Lemma 6.1 and Theorem 4.7. It suffices to show (2) $\Rightarrow$ (1).

By way of contradiction. Assume that there exists an infinite descending chain of  $Ide(L)$ , as follows:

$$I_1 \supset I_2 \supset \dots \supset I_n \supset \dots$$

Pick  $a_i \in I_i \setminus I_{i+1}$  for  $i = 1, 2, \dots, n, \dots$ . Let  $Q_i$  be a value of  $a_i$  with  $Q_i \supseteq I_{i+1}$ .

Now, let  $\{a_1, a_2, \dots, a_n\}$  be a finite basis of  $L$ . We claim that  $L$  has only  $n$  minimal prime ideals  $a_1^\perp, a_2^\perp, \dots, a_n^\perp$ . In fact, it suffices to show that for any  $P \in MinSpe(L)$ , there exists some  $i$  ( $1 \leq i \leq n$ ) such that  $P = a_i^\perp$ . Suppose that

$a_i^\perp \not\subseteq P$  for any  $i$ . Pick  $b_i \in a_i^\perp \setminus P$  for  $i = 1, 2, \dots, n$ , and set  $b = b_1 \wedge b_2 \wedge \dots \wedge b_n$ . Then  $0 < b \notin P$  and  $b \wedge a_i = 0$  for  $i = 1, 2, \dots, n$ , which contradicts the fact that  $\{a_1, a_2, \dots, a_n\}$  is a basis of  $L$ . So  $L$  has only  $n$  minimal prime ideals, written  $P_1, P_2, \dots, P_n$ . Notice that

$$I_2 \subseteq Q_1, I_3 \subseteq Q_2, \dots, I_{n+1} \subseteq Q_n, \dots$$

Then  $L$  has at least infinite many distinct values  $Q_1, Q_2, \dots, Q_n, \dots$ .

For  $P_1$ , since  $L \in \mathbb{E}$ , there exists a finite subset of the set  $\{Q_1, Q_2, \dots, Q_n, \dots\}$  containing  $P_1$  which is a proper descending chain of  $V(L)$ . Now, we omit this subset from the set  $\{Q_1, Q_2, \dots, Q_n, \dots\}$ .

For  $P_2$ , similarly, there also exists a finite subset of the set  $\{Q_1, Q_2, \dots, Q_n, \dots\}$  containing  $P_2$  which is a proper descending chain of  $V(L)$ . We also omit this subset from the set  $\{Q_1, Q_2, \dots, Q_n, \dots\}$ .

Continuing this process, finally, for  $P_n$ , there also exists a finite subset of the set  $\{Q_1, Q_2, \dots, Q_n, \dots\}$  containing  $P_n$  which is a proper descending chain of  $V(L)$ . We similarly omit this subset from the set  $\{Q_1, Q_2, \dots, Q_n, \dots\}$ .

Notice that the set  $\{Q_1, Q_2, \dots, Q_n, \dots\}$  is infinite, the remains are also an infinite subset of  $\{Q_1, Q_2, \dots, Q_n, \dots\}$ , and each of which does not contain any one of minimal prime ideals  $P_1, P_2, \dots, P_n$ . This is clearly impossible. So  $L \in \mathbb{D}$ .  $\square$

## 7. $\mathbb{F}_v$ and $\mathbb{F}$

In this section, we investigate decomposable lattices in which each nonzero element has only finitely many values and decomposable lattices in which each disjoint subset with upper bound is finite.

The following is well known ([10], Theorem 5.9).

**Lemma 7.1.** Let  $L \in \mathbb{DL}$ . The following conditions are equivalent:

- (1)  $L \in \mathbb{F}_v$ .
- (2) For any  $0 < a \in L$ ,  $a = a_1 \vee a_2 \vee \dots \vee a_n$ , where  $a_i \wedge a_j = 0$  for  $i \neq j$  and each  $a_i$  is special.

**Theorem 7.2.** Let  $L \in \mathbb{DL}$ . If  $L \in \mathbb{F}$ , then  $L \in \mathbb{A}$  and  $V(L) = S(L)$ .

**Proof.** We first claim that  $L$  has a basis. Otherwise, there exists  $0 < a \in L$  and  $0 < a_1, a_2 < a$  such that  $a_1 \wedge a_2 = 0$ .  $a$  is not a basic element implies that  $a_1$  is



not also a basic element, then there exist  $0 < a_{11}, a_{12} < a_1$  such that  $a_{11} \wedge a_{12} = 0$ . Continuing this process, we can obtain an infinite disjoint subset  $\{x_1, x_2, \dots, x_n, \dots\}$  of  $L$  with upper bound  $a$ , which contradicts  $L \in \mathbb{F}$ . So  $L$  has a basis.

Now, let  $\{a_\lambda \mid \lambda \in \Lambda\}$  be a basis of  $L$ . We further claim that there exists  $\lambda \in \Lambda$  such that  $a_\lambda^\perp \subseteq P$  for any  $P \in \text{MinSpe}(L)$ . Suppose that  $a_\lambda^\perp \not\subseteq P$  for any  $\lambda \in \Lambda$ . We shall divide the proof into two steps:

**Step 1.** If  $|\Lambda| = \infty$ , then pick a fixed  $\lambda_1 \in \Lambda$ . Since  $a_{\lambda_1}^\perp \not\subseteq P$ , we may further pick  $0 < b_1 \in a_{\lambda_1}^\perp \setminus P$ . Since  $L \in \mathbb{F}$ , there exists a finite subset of  $\Lambda$ , write  $\{\lambda_2, \lambda_3, \dots, \lambda_n\}$ , such that  $b_1 \wedge a_{\lambda_i} > 0$  for  $i = 2, 3, \dots, n$ , and  $b_1 \wedge a_\lambda = 0$  for any  $\lambda \in \Lambda \setminus \{\lambda_2, \lambda_3, \dots, \lambda_n\}$ . Now, pick  $b_i \in a_{\lambda_i}^\perp \setminus P$  for  $i = 2, 3, \dots, n$ , and set  $b = b_1 \wedge b_2 \wedge \dots \wedge b_n$ . Then  $0 < b \notin P$  and  $b \wedge a_\lambda = 0$  for any  $\lambda \in \Lambda$ , which contradicts the fact that  $\{a_\lambda \mid \lambda \in \Lambda\}$  is a basis of  $L$ .

**Step 2.** If  $|\Lambda| = k < \infty$ , then pick  $b_i \in a_{\lambda_i}^\perp \setminus P$  for  $i = 1, 2, \dots, k$ , and set  $b = b_1 \wedge b_2 \wedge \dots \wedge b_k$ . Then  $0 < b \notin P$  and  $b \wedge a_i = 0$  for  $i = 1, 2, \dots, k$ , which also contradicts the fact that  $\{a_1, a_2, \dots, a_k\}$  is a basis of  $L$ .

In view of the above arguments, there exists some  $\lambda \in \Lambda$  such that  $a_\lambda^\perp \subseteq P$ . By Lemma 4.1,  $a_\lambda^\perp \in \text{MinSpe}(L)$ , so that  $P = a_\lambda^\perp$ .

We now show that  $L \in \mathbb{A}$ . Given any  $M \in \text{Spe}(L)$ , there exists some  $P \in \text{MinSpe}(L)$  such that  $M \supseteq P = a^\perp$ . By Lemma 4.1,  $a$  is a basic element, and hence  $a^\perp$  is a maximal poplar. So  $M \supseteq a^\perp$ , which implies that  $M^{\perp\perp} \supseteq a^\perp$ , so that  $a^\perp = M^{\perp\perp}$ . So  $M = a^\perp \in \text{MinSpe}(L)$ . Therefore  $L \in \mathbb{A}$ .

Finally, we show that  $S(L) = V(L)$ . For any  $Q \in V(L)$ , using the above result,  $Q$  is a minimal prime ideal of  $L$  and  $Q = s^\perp$ , where  $s$  is a basic element of  $L$ . So  $Q$  is the unique value of  $s$ , and hence  $S(L) = V(L)$ .  $\square$

**Theorem 7.3.** Let  $L \in \mathbb{DL}$ . If  $A \in \mathbb{A} \cap \mathbb{F}_v$  then  $L \in \mathbb{F}$ .

**Proof.** Assume that  $L \notin \mathbb{F}$ . Then there exists some  $0 < a \in L$  and an infinite disjoint subset  $\{a_\lambda \mid \lambda \in \Lambda\}$  of  $L$  with upper bound  $a$ . Now, let  $Q_\lambda$  be a value of  $a_\lambda$  for any  $\lambda \in \Lambda$ . Since  $L \in \mathbb{F}_v$ ,  $a$  has only finitely many values, write  $Q_1, Q_2, \dots, Q_n$ . Since  $a_\lambda \notin Q_\lambda$  for any  $\lambda \in \Lambda$  implies that  $a \notin Q_\lambda$  for any  $\lambda \in \Lambda$ . This means that for any  $\lambda \in \Lambda$ ,  $Q_\lambda \subseteq Q_i$  for some  $i$  ( $i = 1, 2, \dots, n$ ). Since  $L \in \mathbb{A}$ , we get that for any  $\lambda \in \Lambda$ ,  $Q_\lambda = Q_i$  for some  $i$  ( $i = 1, 2, \dots, n$ ). Notice that  $\Lambda$  is infinite, this is impossible. So  $L \in \mathbb{F}$ .  $\square$

At the end of this paper, we investigate consistency of decomposable lattices in the sense of the following definition.

**Definition 7.4.** Let  $L$  be a lattice. For any  $0 < x \in L$ , we denote by  $v(x)$  the cardinal number of the set of all values of  $x$ , i.e.,  $v(x) = |\text{Val}(x)|$ . A lattice  $L$  is called consistent if  $x \leq y$  then  $v(x) \leq v(y)$ , where  $0 < x, y \in L$ .

According to Definition 7.4, one can obtain that for a decomposable lattice  $L$ , if  $L$  is consistent then  $a$  is special if and only if it is a basic element. In order to investigate the structure of decomposable lattices with consistency, let us recall that a lattice  $L$  is called projectable if for any  $a \in L$ ,  $L = a^{\perp\perp} \vee a^\perp$ .

**Theorem 7.5.** Let  $L \in \mathbb{DL}$ . If  $L \in \mathbb{F}_v$  then the following conditions are equivalent:

- (1)  $L \in \mathbb{T}$ .
- (2)  $L \in \mathbb{B}$ .
- (2)  $L$  is consistent.

**Proof.** (1) $\Rightarrow$ (2) Suppose that  $L \notin \mathbb{B}$ . Then there exist some  $P \in \text{Spe}(L)$  and  $M_1, M_2 \in \text{MinSpe}(L)$  with  $M_1 \neq M_2$  such that  $M_1 \vee M_2 \subseteq P$ . Pick  $a \in M_1 \setminus M_2$ . By Lemma 4.2,  $a^{\perp\perp} \subseteq M_1$  and  $a^\perp \subseteq M_2$ . Then

$$L = a^{\perp\perp} \vee a^\perp \subseteq M_1 \vee M_2 \subseteq P,$$

a contradiction. Thus  $L \in \mathbb{B}$ .

(2) $\Rightarrow$ (3) Given  $x, y \in L$  with  $0 < x \leq y$ , let  $Q_x$  be a value of  $x$ . Then  $x \notin Q_x$  implies that  $y \notin Q_x$ , and hence there exists some  $Q_y \in \text{Val}(y)$  such that  $Q_x \subseteq Q_y$ . In order to show that  $v(x) \leq v(y)$ , it suffices to show that if  $Q_1$  and  $Q_2$  are two distinct values of  $x$ , then  $Q_1$  and  $Q_2$  can not be contained in the same value  $Q$  of  $y$ . Otherwise, since  $L \in \mathbb{B}$ ,  $L = Q_1 \vee Q_2 \subseteq Q$ , a contradiction. Therefore  $L$  is consistent.

(3) $\Rightarrow$ (1) Since  $L$  is consistent, each special element in  $L$  is a basic element. Again, since  $L \in \mathbb{F}_v$ , by Lemma 7.1,  $L$  has a basis. Now, let  $\{a_\lambda \mid \lambda \in \Lambda\}$  be a basis of  $L$ . Then, by Lemma 4.1,  $\{a_\lambda^{\perp\perp} \mid \lambda \in \Lambda\}$  is a set of maximal totally ordered ideals of  $L$ .

**Claim 1.** If  $D$  is a maximal totally ordered ideal of  $L$ , then  $D = a_{\lambda_0}^{\perp\perp}$  for some  $\lambda_0 \in \Lambda$ .

Notice that  $\{a_\lambda \mid \lambda \in \Lambda\}$  is a basis of  $L$ . We can obtain that  $D \cap a_{\lambda_0}^{\perp\perp} \neq 0$  for some  $\lambda_0 \in \Lambda$ . Otherwise,  $D \cap a_\lambda^{\perp\perp} = 0$  for any  $\lambda \in \Lambda$  implies that  $d \wedge a_\lambda = 0$  for any

$\lambda \in \Lambda$  and  $0 < d \in D$ , which contradicts the fact that  $\{a_\lambda \mid \lambda \in \Lambda\}$  is a basis of  $L$ . So  $D = a_{\lambda_0}^{\perp\perp}$  for some  $\lambda_0 \in \Lambda$ .

**Claim 2.** For any  $0 < g \in L$ ,  $g^{\perp\perp} = \bigvee_{\lambda_i \in \Lambda_1} a_{\lambda_i}^{\perp\perp}$ , where  $\Lambda_1$  is a finite subset of  $\Lambda$ .

Since  $L \in \mathbb{F}_v$ , by Lemma 7.1,  $g = g_1 \vee g_2 \vee \cdots \vee g_n$ , where  $g_i \wedge g_j = 0$  for  $i \neq j$  and each  $g_i$  is special. Moreover, each  $g_i$  is a basic element. Using Claim 1, each  $g_i \in g_i^{\perp\perp} = a_{\lambda_i}^{\perp\perp}$  for some  $\lambda_i \in \Lambda$ , where  $i = 1, 2, \dots, n$ . Set

$$\Lambda_1 = \{\lambda_1, \lambda_2, \dots, \lambda_n\} \text{ and } \Lambda_2 = \Lambda \setminus \Lambda_1.$$

Clearly,  $g^{\perp\perp} = \bigvee_{\lambda_i \in \Lambda_1} a_{\lambda_i}^{\perp\perp}$ .

**Claim 3.** For any  $0 < g \in L$ ,  $g^\perp = \bigvee_{\lambda \in \Lambda_2} a_\lambda^{\perp\perp}$ .

For any  $\lambda \in \Lambda_2$ ,  $a_\lambda \wedge a_{\lambda_i} = 0$  for any  $\lambda_i \in \Lambda_1$ , which implies that  $a_\lambda \wedge g_i = 0$  for  $i = 1, 2, \dots, n$ . So  $a_\lambda \wedge g = 0$ . Then  $a_\lambda \in g^\perp$  for any  $\lambda \in \Lambda_2$ , so that  $a_\lambda^{\perp\perp} \subseteq g^\perp$  for any  $\lambda \in \Lambda_2$ . So  $\bigvee_{\lambda \in \Lambda_2} a_\lambda^{\perp\perp} \subseteq g^\perp$ .

Conversely, for any  $0 < h \in g^\perp$ , since  $L \in \mathbb{F}_v$ , write  $h = h_1 \vee h_2 \vee \cdots \vee h_m$ , where  $h_i \wedge h_j = 0$  for  $i \neq j$  and each  $h_j$  is special. So each  $h_j$  is a basic element. Since  $h \wedge g = 0$  implies that  $h \wedge g_i = 0$  for  $i = 1, 2, \dots, n$ , so that  $h_j \wedge g_i = 0$  for  $j = 1, 2, \dots, m$ . From this we can further obtain that  $h_j \wedge a_{\lambda_i} = 0$  for any  $\lambda_i \in \Lambda_1$  and  $j = 1, 2, \dots, m$ . Using Claim 1, each  $h_j \in h_j^{\perp\perp} = a_{\lambda_k}^{\perp\perp}$  for some  $\lambda_k \in \Lambda_2$  and  $j = 1, 2, \dots, m$ . So

$$h \in \bigvee_{j=1}^m h_j^{\perp\perp} = \bigvee_{k=1}^m a_{\lambda_k}^{\perp\perp} \subseteq \bigvee_{\lambda \in \Lambda_2} a_\lambda^{\perp\perp}.$$

Thus  $g^\perp \subseteq \bigvee_{\lambda \in \Lambda_2} a_\lambda^{\perp\perp}$ . Therefore  $g^\perp = \bigvee_{\lambda \in \Lambda_2} a_\lambda^{\perp\perp}$ .

Finally, since

$$L = \bigvee_{\lambda \in \Lambda} a_\lambda^{\perp\perp},$$

using the results of Claim 2 and Claim 3, we then have

$$L = \bigvee_{\lambda \in \Lambda} a_\lambda^{\perp\perp} = \left( \bigvee_{\lambda_i \in \Lambda_1} a_{\lambda_i}^{\perp\perp} \right) \vee \left( \bigvee_{\lambda \in \Lambda_2} a_\lambda^{\perp\perp} \right) = g^{\perp\perp} \vee g^\perp.$$

Therefore  $L \in \mathbb{T}$ . □

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